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Steady-state response of axially moving viscoelastic beams with pulsating speed: comparison of two nonlinear models

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Abstract

Principal parametric resonance in transverse vibration is investigated for viscoelastic beams moving with axial pulsating speed. A nonlinear partial-differential equation governing the transverse vibration is derived from the dynamical, constitutive, and geometrical relations. Under certain assumption, the partial-differential reduces to an integro-partial-differential equation for transverse vibration of axially accelerating viscoelastic nonlinear beams. The method of multiple scales is applied to two equations to calculate the steady-state response. Closed form solutions for the amplitude of the vibration are derived from the solvability condition of eliminating secular terms. The stability of straight equilibrium and nontrivial steady-state response are analyzed by use of the Lyapunov linearized stability theory. Numerical examples are presented to highlight the effects of speed pulsation, viscoelascity, and nonlinearity and to compare results obtained from two equations.

Keywords: Axially accelerating beam; Principal parametric resonance; Nonlinearity; Viscoelascity

1. Introduction

Many engineering devices involve the transverse vibrations of axially moving materials. One major problem is the occurrence of large transverse vibrations, termed as parametric vibrations, due to initial tension or axial speed variations (Wickert and Mote, 1988; Abrate, 1992; Chen, in press). In fact, many real systems can be represented by the axially moving materials with pulsating speed. That is, the axial transport speed is a constant mean velocity with small periodic fluctuations. There are comprehensive studies on such systems.

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Pasin (1972) studied the stability of transverse vibrations of beams with periodically reciprocating motion in axial direction. Pakdemirli et al. (1994) investigated dynamic stability in transverse vibration of an axially accelerating string based on the Galerkin truncation. Pakdemirli and Ulsoy (1997) applied the discretization–perturbation method and the direct-perturbation method to analyze the stability of an axially accelerating string. Öz et al. (1998) applied the method of multiple scales to study dynamic stability of an axially accelerating beam with small bending stiffness. Based on one-term Galerkin discretization, Ravindra and Zhu (1998) analyzed chaotic behaviors of axially accelerating beams. Özkaya and Pakdemirli (2000) applied the method of multiple scales and the method of matched asymptotic expansions to construct nonresonant boundary layer solutions for an axially accelerating beam with small bending stiffness. Öz and Pakdemirli (1999) and Öz (2001) used the method of multiple scales to calculate analytically the stability boundaries of an axially accelerating beam under simply supported conditions and fixed–fixed conditions, respectively. Parker and Lin (2001) adopted a one-term Galerkin discretization and the perturbation method to study dynamic stability of an axially accelerating beam subjected to a tension fluctuation. Öz et al. (2001) applied the method of multiple scales to determine the steady-state transverse response and its stability of axially accelerating nonlinear beams. Özkaya and Öz (2002) applied artificial neural network algorithm to determine stability of an axially accelerating beam.

All above-mentioned researchers assumed the strings or beams under their consideration to be elastic, and did not account for any dissipative mechanisms. Nevertheless, the modeling of dissipative mechanisms is an important research topic of axially moving material vibrations (Wickert and Mote, 1988; Abrate, 1992; Chen, in press). Viscoelascity is an effective approach to model the damping mechanism (Park, 2001). The parametric vibrations of axially moving viscoelastic strings have been extensively investigated. These researches include numerical simulation of transient vibrations (Fung et al., 1997, 1998; Zhao and Chen, 2002; Chen and Zhao, in press; Chen et al., in press), analytical expressions of steady-state responses (Zhang and Zu, 1999a,b; Chen and Zu, 2003; Chen et al., 2003, 2004d), and chaotic behaviors (Chen et al., 2003, 2004a,b; Chen and Zhang, 2004). However, The literature that is specially related to axially accelerating viscoelastic beams is relatively limited. Based on three-term Galerkin discretization, Marynowski (2002) and Marynowski and Kapitaniak (2002) compared the Kelvin model with the Maxwell model and the Bügers model, respectively through numerical simulation of nonlinear vibration responses of an axially moving beam excited by a changing tension, and found that all models yield similar results in the case of small damping. Marynowski (2004) further studied numerically nonlinear dynamical behavior of an axially moving viscoelastic beam with time-dependent tension based on four-term Galerkin discretization. Based on second-term Galerkin discretization, Chen et al. (2004c) analyzed the stability of axially accelerating linear beams, and Yang and Chen (2005) studied numerically bifurcation and chaos of an axially accelerating nonlinear beam. So far the analytical studies on axially accelerating viscoelastic beams have been confined to the linear model, and only numerical approaches have been used to treat nonlinear vibration of axially accelerating viscoelastic beams. To address the lack of research in this aspect, the present investigation is devoted to analytical study of axially accelerating viscoelastic nonlinear beams.

When transverse motion is treated for axially moving beams, there are two types of nonlinear models, a partial-differential equation or an integro-partial-differential equation. The partial-differential equation is derived from considering the transverse displacement only, and the integro-partial-differential equation is traditionally derived from decoupling the governing equation of coupled longitudinal and transverse motion under the quasi-static stretch assumption that supposes that the influence of longitudinal inertia can be neglected. The application of the partial-differential equation is limited (Marynowski, 2002, 2004; Marynowski and Kapitaniak, 2002; Yang and Chen, 2005), while the quasi-static stretch assumption is widely adopted in parametric vibration (Ravindra and Zhu, 1998; Chakraborty and Mallik, 1998; Öz et al., 2001; Pellicano et al., 2001) as well as free vibration (Wickert, 1991; Pellicano and Zirilli, 1997; Pellicano and Vestroni, 2000) and forced vibration (Pellicano and Vestroni, 2002; Chakraborty et al., 1999). In present investigation, two nonlinear models are developed for transverse vibration of an axially accelerating viscoelastic beam. The method of multiple scales is applied to both models without discretization. The corresponding steady-state responses and their stability are compared.

2. Governing equations of an axially accelerating viscoelastic beam

A uniform axially moving viscoelastic beam, with density ρ, cross-sectional area A, moment of inertial I, and initial tension P_0, travels at the time-dependent axial transport speed $v(T)$ between two prismatic ends separated by distance L. Consider only the bending vibration described by the transverse displacement $U(X,T)$, where T is the time and X is the axial coordinate. The Newton's second law of motion yields

$$\rho A\left(\frac{\partial^2 U}{\partial T^2}+2v\frac{\partial^2 U}{\partial X\partial T}+\frac{\mathrm{d}v}{\mathrm{d}T}\frac{\partial U}{\partial X}+v^2\frac{\partial^2 U}{\partial X^2}\right)=\frac{\partial}{\partial X}\left[(P_0+A\sigma)\frac{\partial U}{\partial X}\right]-\frac{\partial^2 M(X,T)}{\partial X^2} \tag{1}$$

where $\sigma(X,T)$ and $\varepsilon_L(X,T)$ are respectively the axial disturbed stress and $M(X,T)$ is the bending moment. The viscoelastic material of the beam obeys the Kelvin model, with the constitution relation

$$\sigma(X,T)=E\varepsilon_{\mathrm{L}}(X,T)+\eta\frac{\partial\varepsilon_{\mathrm{L}}(X,T)}{\partial T} \tag{2}$$

where $\varepsilon_{\mathrm{L}}(X,T)$ is the Lagrangian strain

$$\varepsilon_{\mathrm{L}}(X,T)=\frac{1}{2}\left[\frac{\partial U(X,T)}{\partial X}\right]^2 \tag{3}$$

which is used to account for geometric nonlinearity due to small but finite stretching of the beam.

For a slender beam (for example with $I/(AL^2)<0.001$), the linear moment–curvature relationship is sufficiently accurate,

$$M(X,T)=EI\frac{\partial^2 U(X,T)}{\partial X^2}+\eta I\frac{\partial^3 U(X,T)}{\partial X^2\partial T} \tag{4}$$

Substitution of Eqs. (2)–(4) into Eq. (1) yields the governing equation of transverse motion of an axially accelerating viscoelastic beam

$$\begin{aligned}&\rho A\left(\frac{\partial^2 U}{\partial T^2}+2v\frac{\partial^2 U}{\partial X\partial T}+\frac{\mathrm{d}v}{\mathrm{d}T}\frac{\partial U}{\partial X}+v^2\frac{\partial^2 U}{\partial X^2}\right)-P_0\frac{\partial^2 U}{\partial X^2}+EI\frac{\partial^4 U}{\partial X^4}+\eta I\frac{\partial^5 U}{\partial T\partial X^4}\\&=\frac{3}{2}E\left(\frac{\partial U}{\partial X}\right)^2\frac{\partial^2 U}{\partial X^2}+2\eta\frac{\partial U}{\partial X}\frac{\partial^2 U}{\partial X\partial T}\frac{\partial^2 U}{\partial X^2}+\eta\left(\frac{\partial U}{\partial X}\right)^2\frac{\partial^3 U}{\partial X^2\partial T}\end{aligned} \tag{5}$$

If the spatial variation of the tension is rather small, then one can use the averaged value of the disturbed tension $\frac{1}{L}\int_0^L A\sigma\,\mathrm{d}x$ to replace the exact value $A\sigma$. In this case, Eq. (1) becomes

$$\rho A\left(\frac{\partial^2 U}{\partial T^2}+2v\frac{\partial^2 U}{\partial X\partial T}+\frac{\mathrm{d}v}{\mathrm{d}T}\frac{\partial U}{\partial X}+v^2\frac{\partial^2 U}{\partial X^2}\right)=\frac{\partial}{\partial X}\left[\left(P_0+\frac{1}{L}\int_0^L A\sigma\,\mathrm{d}x\right)\frac{\partial U}{\partial X}\right]-\frac{\partial^2 M(X,T)}{\partial X^2} \tag{6}$$

Substitution of Eqs. (2)–(4) into Eq. (6) leads to

$$\begin{aligned}&\rho A\left(\frac{\partial^2 U}{\partial T^2}+2v\frac{\partial^2 U}{\partial X\partial T}+\frac{\mathrm{d}v}{\mathrm{d}T}\frac{\partial U}{\partial X}+v^2\frac{\partial^2 U}{\partial X^2}\right)-P_0\frac{\partial^2 U}{\partial X^2}+EI\frac{\partial^4 U}{\partial X^4}+\eta I\frac{\partial^5 U}{\partial T\partial X^4}\\&=\frac{A}{L}\int_0^L\left[\frac{E}{2}\left(\frac{\partial U}{\partial X}\right)^2+\eta\frac{\partial U}{\partial X}\frac{\partial^2 U}{\partial X\partial T}\right]\mathrm{d}x\frac{\partial^2 U}{\partial X^2}\end{aligned} \tag{7}$$

Obviously, Eq. (6) is a nonlinear partial-differential equation, while Eq. (7) is a integro-partial-differential equation.

Eq. (5) can be derived from the governing equation for coupled longitudinal and transverse vibration from considering the transverse vibration only and setting all longitudinal variables to zero. Eq. (7) can be obtained through uncoupling the governing equation for coupled longitudinal and transverse vibration under the quasi-static stretch assumption in a similar way in (Wickert, 1992). Under this assumption, the dynamic tension component to be a function of time alone. In traditional derivation, Eq. (7) seems more exact than Eq. (5) because it is the transverse equation of motion in which the longitudinal displacement field is taken into account. However, the derivation here indicates that Eq. (5) can be reduced to Eq. (7) based on the quasi-static stretch assumption.

3. The multiscale analysis

Introduce the dimensionless variables and parameters

$$u=\frac{U}{\sqrt{\varepsilon}L},\quad x=\frac{X}{L},\quad t=T\sqrt{\frac{P_0}{\rho AL^2}},\quad \gamma=v\sqrt{\frac{\rho A}{P_0}},\quad k_{\mathrm{f}}^2=\frac{EI}{P_0L^2},\quad \alpha=\frac{I\eta}{\varepsilon L^3\sqrt{\rho AP_0}}$$
$$k_1=\sqrt{\frac{EA}{P_0}},\quad k_2=\frac{A\eta}{\varepsilon L\sqrt{P_0\rho A}} \tag{8}$$

where bookkeeping device ε is a small dimensionless parameter accounting for the fact that both the transverse displacement and the viscosity coefficient are very small. Eqs. (5) and (7) can be respectively cast into the dimensionless form

$$\frac{\partial^2 u}{\partial t^2}+2\gamma\frac{\partial^2 u}{\partial x\partial t}+\frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\partial u}{\partial x}+(\gamma^2-1)\frac{\partial^2 u}{\partial x^2}+k_{\mathrm{f}}^2\frac{\partial^4 u}{\partial x^4}+\varepsilon\alpha\frac{\partial^5 u}{\partial x^4\partial t}$$
$$=\frac{3}{2}\varepsilon k_1^2\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)^2+\varepsilon^2\alpha k_2\left[2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial x^2\partial t}\right] \tag{9}$$

$$\frac{\partial^2 u}{\partial t^2}+2\gamma\frac{\partial^2 u}{\partial x\partial t}+\frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\partial u}{\partial x}+(\gamma^2-1)\frac{\partial^2 u}{\partial x^2}+k_{\mathrm{f}}^2\frac{\partial^4 u}{\partial x^4}+\varepsilon\alpha\frac{\partial^5 u}{\partial x^4\partial t}$$
$$=\frac{\partial^2 u}{\partial x^2}\int_0^1\left[\frac{1}{2}\varepsilon k_1^2\left(\frac{\partial u}{\partial x}\right)^2+\varepsilon^2\alpha k_2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\right]\mathrm{d}x \tag{10}$$

In the present investigation, the axial speed is assumed to be a small simple harmonic variation about the constant mean speed,

$$\gamma(t)=\gamma_0+\varepsilon\gamma_1\sin\omega t \tag{11}$$

Substitution of Eq. (11) into Eqs. (9) and (10), respectively yields

$$M\frac{\partial^2 u}{\partial t^2}+G\frac{\partial u}{\partial t}+Ku$$
$$=\frac{3}{2}\varepsilon k_1^2\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)^2-2\varepsilon\gamma_1\sin\omega t\frac{\partial^2 u}{\partial x\partial t}-2\varepsilon\gamma_0\gamma_1\sin\omega t\frac{\partial^2 u}{\partial x^2}-\varepsilon\omega\gamma_1\cos\omega t\frac{\partial u}{\partial x}-\varepsilon\alpha\frac{\partial^5 u}{\partial x^4\partial t}+\mathrm{O}(\varepsilon^2) \tag{12}$$

$$M\frac{\partial^2 u}{\partial t^2}+G\frac{\partial u}{\partial t}+Ku$$
$$=\frac{1}{2}\varepsilon k_1^2\frac{\partial^2 u}{\partial x^2}\int_0^1\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x-2\varepsilon\gamma_1\sin\omega t\frac{\partial^2 u}{\partial x\partial t}-2\varepsilon\gamma_0\gamma_1\sin\omega t\frac{\partial^2 u}{\partial x^2}-\varepsilon\omega\gamma_1\cos\omega t\frac{\partial u}{\partial x}-\varepsilon\alpha\frac{\partial^5 u}{\partial x^4\partial t}+\mathrm{O}(\varepsilon^2) \tag{13}$$

where the mass, gyroscopic, and linear stiffness operators are respectively defined as

$$M=I,\quad G=2\gamma_0\frac{\partial}{\partial x},\quad K=(\gamma_0^2-1)\frac{\partial^2}{\partial x^2}+v_\mathrm{f}^2\frac{\partial^4}{\partial x^4} \tag{14}$$

The method of multiple scales will be employed to solve Eqs. (12). A first order uniform approximation is sought in the form

$$u(x,t;\varepsilon)=u_0(x,T_0,T_1)+\varepsilon u_1(x,T_0,T_1)+\mathrm{O}(\varepsilon^2) \tag{15}$$

where $T_0=\tau$ is a fast scale characterizing motions occurring at ω_k (one of the natural frequencies of the corresponding unperturbed linear system), and $T_1=\varepsilon\tau$ is a slow scale characterizing the modulation of the amplitudes and phases due to nonlinearity, viscoelascity, and possible resonance. Substitution of Eq. (15) and the following relationship

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\mathrm{O}(\varepsilon^2),\quad \frac{\partial^2}{\partial t^2}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1}+\mathrm{O}(\varepsilon^2) \tag{16}$$

into Eq. (9) and then equalization of coefficients of ε^0 and ε in the resulting equation lead to

$$M\frac{\partial^2 u_0}{\partial T_0^2}+G\frac{\partial u_0}{\partial T_0}+Ku_0=0 \tag{17}$$

$$M\frac{\partial^2 u_1}{\partial T_0^2}+G\frac{\partial u_1}{\partial T_0}+Ku_1=\frac{3}{2}k_1^2\frac{\partial^2 u_0}{\partial x^2}\left(\frac{\partial u_0}{\partial x}\right)-2\frac{\partial^2 u_0}{\partial T_0\partial T_1}-2\gamma_0\frac{\partial^2 u_0}{\partial x\partial T_1}$$
$$-2\gamma_1\sin\omega t\left(\frac{\partial^2 u_0}{\partial x\partial T_0}+\gamma_0\frac{\partial^2 u_0}{\partial x^2}\right)-\gamma_1\omega\cos\omega t\frac{\partial u_0}{\partial x}-\alpha\frac{\partial^5 u_0}{\partial x^4\partial T_0} \tag{18}$$

Under the simple support boundary conditions

$$u(0,t)=u(1,t)=0,\quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1}=0 \tag{19}$$

The solution to Eq. (17) has been given by Wickert and Mote (1990)

$$u_0(x,T_0,T_1)=\sum_{n=1}^{\infty}[\phi_n(x)A_n(T_1)e^{\mathrm{i}\omega_n T_0}+\bar{\phi}_n(x)\bar{A}_n(T_1)\mathrm{e}^{-\mathrm{i}\omega_n T_0}] \tag{20}$$

where ω_n and ϕ_n are respectively the nth natural frequency and complex mode function of the corresponding linear homogeneous system. Under the boundary conditions (19), the mode function is (Öz and Pakdemirli, 1999)

$$\phi_n(x) = e^{i\beta_n x} - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{3n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{2n}^2)(e^{i\beta_{3n}} - e^{i\beta_{2n}})} e^{i\beta_{2n}x} - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{3n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{3n}^2)(e^{i\beta_{3n}} - e^{i\beta_{3n}})} e^{i\beta_{2n}x}$$
$$- \left[1 - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{3n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{2n}^2)(e^{i\beta_{3n}} - e^{i\beta_{2n}})} - \frac{(\beta_{4n}^2 - \beta_{1n}^2)(e^{i\beta_{3n}} - e^{i\beta_{1n}})}{(\beta_{4n}^2 - \beta_{3n}^2)(e^{i\beta_{3n}} - e^{i\beta_{3n}})}\right] e^{i\beta_{4n}x} \tag{21}$$

where β_{jn} $(j = 1,2,3,4)$ are four roots of the following four-order algebraic equation

$$k_f^4\beta_{jn}^4 + (1-\gamma^2)\beta_{jn}^2 - 2\omega_n\beta_{jn} - \omega_n^2 = 0 \tag{22}$$

If the variation frequency ω approaches two times of any natural frequency of the system (17), principal parametric resonance may occur. To explore the nth principal resonance, it does not lose generality for u_0 to include only the nth mode vibration

$$u_0(x, T_0, T_1) = \phi_n(x)A_n(T_1)e^{i\omega_n T_0} + \text{cc} \tag{23}$$

where cc stands for the complex conjugate of all preceding terms on the right hand of an equation. Substituting Eq. (23) into Eq. (18) and expressing the trigonometric functions in exponential form yield

$$M\frac{\partial^2 u_1}{\partial T_0^2} + G\frac{\partial u_1}{\partial T_0} + Ku_1$$
$$= -2i\omega_n\dot{A}_n\phi_n e^{i\omega_n T_0} - 2\gamma_0\dot{A}_n\phi_n' e^{i\omega_n T_0} - \alpha i\omega_n A_n\phi_n'''' e^{i\omega_n T_0} + \left(\gamma_1\omega_n\bar{A}_n\bar{\phi}_n' + i\gamma_0\gamma_1\bar{A}_n\bar{\phi}_n'' - \gamma_1\frac{\omega}{2}\bar{A}_n\bar{\phi}_n'\right)e^{i(\omega-\omega_n)T_0}$$
$$+ \left(-\gamma_1\omega_n A_n\phi_n' + i\gamma_0\gamma_1 A_n\phi_n'' - \gamma_1\frac{\omega}{2}A_n\phi_n'\right)e^{i(\omega+\omega_n)T_0}$$
$$+ k_1^2 A_n^2\bar{A}_n\left(\frac{3}{2}\bar{\phi}_n''\phi_n'^2 + 3\phi_n''\phi_n'\bar{\phi}_n'\right)e^{i\omega_n T_0} + \frac{3}{2}k_1^2 A_n^3\phi_n''\phi_{nn}'^2 e^{3i\omega_n T_0} + \text{cc} \tag{24}$$

where the dot and the prime denote derivation with respect to the slow time variable T_1 and the dimensionless spatial variable x, respectively. A detuning parameter σ is introduced to quantify the deviation of ω from $2\omega_n$ and ω is described by

$$\omega = 2\omega_n + \varepsilon\sigma \tag{25}$$

Substitution of Eq. (25) into Eq. (24) leads to

$$M\frac{\partial^2 u_1}{\partial T_0^2} + G\frac{\partial u_1}{\partial T_0} + Ku_1$$
$$= -2i\omega_n\dot{A}_n\phi_n e^{i\omega_n T_0} - 2\gamma_0\dot{A}_n\phi_n' e^{i\omega_n T_0} - \alpha i\omega_n A_n\phi_n'''' e^{i\omega_n T_0} + i\gamma_0\gamma_1\bar{\phi}_n\bar{\phi}_n'' e^{i\omega_n T_0}e^{i\varepsilon\sigma T_0}$$
$$+ (-2\gamma_1\omega_n A_n\phi_n' + i\gamma_0\gamma_1 A_n\phi_n'')e^{3i\omega_n T_0}e^{i\varepsilon\sigma T_0} + \varepsilon\gamma_1\sigma(\bar{A}_n\bar{\phi}_n' e^{i\omega_n T_0} - A_n\phi_n' e^{3i\omega_n T_0})e^{i\varepsilon\sigma T_0}$$
$$+ k_1^2 A_n^2\bar{A}_n\left(\frac{3}{2}\bar{\phi}_n''\phi_n'^2 + 3\phi_n''\phi_n'\bar{\phi}_n'\right)e^{i\omega_n T_0} + \frac{3}{2}k_1^2 A_n^3\phi_n''\phi_{nn}'^2 e^{3i\omega_n T_0} + \text{cc} \tag{26}$$

The terms with ε should be regrouped in the equation of ε^2 order. Hence Eq. (26) can be cast in the form

$$M\frac{\partial^2 u_1}{\partial T_0^2} + G\frac{\partial u_1}{\partial T_0} + Ku_1$$
$$= \left[-2\dot{A}_n(i\omega_n\phi_n + \gamma_0\phi_n') + i\gamma_0\gamma_1\bar{\phi}_n''\bar{\phi}_n e^{i\sigma T_1} - i\alpha\omega_n A_n\phi_n'''' + k_1^2\left(\frac{3}{2}\bar{\phi}_n''\phi_n'^2 + 3\phi_n''\phi_n'\bar{\phi}_n'\right)A_n^2\bar{A}_n\right]e^{i\omega_n T_0} + \text{cc} + \text{NST} \tag{27}$$

where NST stands for the terms that will not bring secular terms into the solution.

Eq. (27) has a bounded solution only if a solvability condition holds. The solvability condition demands that the possible secular term coefficient at right hand of Eq. (27) be orthogonal to every solution of the homogeneous problem. That is

$$\left\langle -2\dot{A}_n(\mathrm{i}\omega_n\phi_n + \gamma_0\phi_n') + \mathrm{i}\gamma_0\gamma_1\bar{\phi}_n''\bar{\phi}_n\mathrm{e}^{\mathrm{i}\sigma T_1} - \mathrm{i}\alpha\omega_n A_n\phi_n'''' + k_1^2\left(\frac{3}{2}\bar{\phi}_n''\phi_n'^2 + 3\phi_n''\phi_n'\bar{\phi}_n'\right)A_n^2\bar{A}_n, \phi_n\right\rangle = 0 \tag{28}$$

where the inner product is defined for complex functions $f(x)$ and $g(x)$ on [0, 1] as

$$\langle f, g\rangle = \int_0^1 f\bar{g}\,\mathrm{d}x \tag{29}$$

Application of the distributive law of the inner product to Eq. (28) leads to

$$\dot{A}_n + \alpha\mu_n A_n + \gamma_1\chi_n\bar{A}_m\mathrm{e}^{\mathrm{i}\sigma T_1} + k_1^2\kappa_n A_n^2\bar{A}_n = 0 \tag{30}$$

where

$$\mu_n = \frac{\mathrm{i}\omega_n\int_0^1\phi_n''''\bar{\phi}_n\mathrm{d}x}{2(\mathrm{i}\omega_n\int_0^1\phi_n\bar{\phi}_n\mathrm{d}x + \gamma_0\int_0^1\phi_n'\bar{\phi}_n\mathrm{d}x)}, \quad \chi_n = -\frac{\mathrm{i}\gamma_0\int_0^1\bar{\phi}_n''\bar{\phi}_n\mathrm{d}x}{2\left(\mathrm{i}\omega_n\int_0^1\phi_n\bar{\phi}_n\mathrm{d}x + \gamma_0\int_0^1\phi_n'\bar{\phi}_n\mathrm{d}x\right)} \tag{31}$$

and

$$\kappa_n = \frac{\frac{3}{2}\int_0^1\bar{\phi}_n\bar{\phi}_n''\bar{\phi}_n'^2\mathrm{d}x + 3\int_0^1\bar{\phi}_n\phi_n''\phi_n'\bar{\phi}_n'\mathrm{d}x}{2\left(\mathrm{i}\omega_n\int_0^1\bar{\phi}_n\phi_n\mathrm{d}x + \gamma_0\int_0^1\bar{\phi}_n\phi_n'\mathrm{d}x\right)} \tag{32}$$

The coefficients μ_n, χ_n, and κ_n are determined by the modal parameters of unperturbed system (17), which are dependent on γ_0 and k_f and are independent of, γ_1 and k_1.

Principal parametric resonance in the system governed by Eq. (13) can be analyzed via the method of multiple scales in a similar way. The solvability condition is still expressed by Eq. (30) with coefficients given by Eq. (31), while the other coefficients is defined by

$$\kappa_n = \frac{\int_0^1\bar{\phi}_n\bar{\phi}_n''\mathrm{d}x\int_0^1\bar{\phi}_n'^2\mathrm{d}x + 2\int_0^1\phi_n'\bar{\phi}_n'\mathrm{d}x\int_0^1\bar{\phi}_n\phi_n''\mathrm{d}x}{4\left(\mathrm{i}\omega_n\int_0^1\bar{\phi}_n\phi_n\mathrm{d}x + \gamma_0\int_0^1\bar{\phi}_n\phi_n'\mathrm{d}x\right)} \tag{33}$$

instead of Eq. (32).

4. Steady-state responses and their stability

Express the solution to Eq. (30) in polar form

$$A_n(T_1) = a_n(T_1)\mathrm{e}^{\mathrm{i}\varphi_n(T_1)} \tag{34}$$

In Eq. (34), $a_n(T_1)$ and $\varphi_n(T_1)$ are respectively the amplitude and the phase angle of the response in the nth principal parametric resonance. For mode functions given by Eq. (21), it can be numerically verified that

$$\mathrm{Re}(\mu_n) > 0,\ \ \mathrm{Im}(\mu_n) = 0; \quad \mathrm{Re}(\kappa_n) = 0,\ \ \mathrm{Im}(\kappa_n) > 0 \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (30) and separating the resulting equation into real and imaginary parts give

$$\begin{aligned} a_n' &= [\alpha \mathrm{Re}(\mu_n) + \gamma_1 \mathrm{Im}(\chi_n)\sin\theta_n - \gamma_1 \mathrm{Re}(\chi_n)\cos\theta_n]a_n \\ a_n\theta_n' &= a_n\sigma + 2\gamma_1[\mathrm{Re}(\chi_n)\sin\theta_n + \mathrm{Im}(\chi_n)\cos\theta_n]a_n - \frac{1}{2}v_1^2\mathrm{Im}(\kappa_n)a_n^3 \end{aligned} \tag{36}$$

where

$$\theta_n = \sigma T_1 - 2\varphi_n \tag{37}$$

For the steady-state response, the amplitude a_n and the new phase angle θ_n in Eq. (36) are constant. Setting $a_n' = 0$ and $\theta_n' = 0$ in Eq. (30), and then eliminating θ_n from the resulting equations lead to

$$\alpha^2[\mathrm{Re}(\mu_n)]^2 + \left[-\frac{\sigma}{2} + \frac{1}{4}k_1^2\mathrm{Im}(\kappa_n)a_n^2\right]^2 = \gamma_1^2[\mathrm{Re}(\chi_n)]^2 + \gamma_1^2[\mathrm{Im}(\chi_n)]^2 \tag{38}$$

It is obvious that Eq. (38) possesses a singular point at the origin (trivial zero solution), which represents the straight equilibrium. In addition, there may exist nontrivial periodic solution with amplitudes defined by Eq. (38), namely

$$a_{n1,2} = \frac{\sqrt{\mathrm{Im}}(\kappa_\mathrm{n})}{k_1}\sqrt{2\sigma \pm 4\sqrt{\gamma_1^2\,|\chi_n|^2 - \alpha^2[\mathrm{Re}(\mu_n)]^2}} \tag{39}$$

Eq. (39) is the closed form solution of the amplitude of nontrivial steady-state response. From Eq. (39), it can be concluded that the nontrivial steady-state solutions exist only if the following conditions hold,

$$\alpha \leqslant \frac{\gamma_1\,|\chi_n|}{\mathrm{Re}(\mu_n)} \tag{40}$$

$$\sigma \geqslant \sigma_{1,2} = \mp 2\sqrt{\gamma_1^2\,|\chi_n|^2 - \alpha^2[\mathrm{Re}(\mu_n)]^2} \tag{41}$$

To determine the stability of the trivial solution, suppose that the perturbed solutions of Eq. (36) take the form

$$A_n(T_1) = \frac{1}{2}[p_n(T_1) + \mathrm{i}q_n(T_1)]\mathrm{e}^{\frac{\sigma T_1}{2}i} \tag{42}$$

where p_n and q_n are real functions. Substituting Eqs. (34) and (35) into Eq. (36) and separating the resulting equation into real and imaginary parts yield

$$\begin{aligned} \dot{p}_n &= -[\alpha\mathrm{Re}(\mu_n) + \gamma_1\mathrm{Re}(\chi_n)]p_n + \left[\frac{\sigma}{2} - \gamma_1\mathrm{Im}(\chi_n)\right]q_n - \frac{1}{4}k_1^2\mathrm{Im}(\kappa_n)(p_n^2+q_n^2)q_n \\ \dot{q}_n &= -\left[\frac{\sigma}{2} + \gamma_1\mathrm{Im}(\chi_n)\right]p_n - [\alpha\mathrm{Re}(\mu_n) - \gamma_1\mathrm{Re}(\chi_n)]q_n + \frac{1}{4}k_1^2\mathrm{Im}(\kappa_n)(p_n^2+q_n^2)p_n \end{aligned} \tag{43}$$

The Jacobian matrix of the right hand function of Eq. (43), calculated at (0, 0) is

$$\begin{pmatrix} -\alpha\mathrm{Re}(\mu_n) - \gamma_1\mathrm{Re}(\chi_n) & \frac{\sigma}{2} - \gamma_1\mathrm{Im}(\chi_n) \\ -\frac{\sigma}{2} - \gamma_1\mathrm{Im}(\chi_n) & -\alpha\mathrm{Re}(\mu_n) + \gamma_1\mathrm{Re}(\chi_n) \end{pmatrix} \tag{44}$$

with the characteristic equation

$$\lambda^2 + 2\alpha\mathrm{Re}(\mu_n)\lambda - \gamma_1^2\,|\chi_n|^2 + \alpha^2[\mathrm{Re}(\mu_n)]^2 + \left(\frac{\sigma}{2}\right)^2 = 0 \tag{45}$$

By the use of the Routh–Hurwitz criterion, the stability conditions can be determined as

$$\sigma < \sigma_1 = -2\sqrt{\gamma_1^2\,|\chi_n|^2 - \alpha^2[\mathrm{Re}(\mu_n)]^2} \tag{46}$$

or

$$\sigma > \sigma_2 = 2\sqrt{\gamma_1^2 \mid \chi_n \mid^2 - \alpha^2 [\mathrm{Re}(\mu_n)]^2} \tag{47}$$

Otherwise, the trivial solution is unstable. The Lyapunov linearized stability theory indicates that the instability of a nonlinear system coincides with that of the corresponding linear system. Hence there exists an instability interval $[\sigma_1, \sigma_2]$ of trivial solution. The instability condition of the trivial solution coincides with the existence condition of the first nontrivial solution, and the stability condition of the trivial solution coincides with the existence condition of the second nontrivial solution.

The stability of the nontrivial solutions can be determined by the following equation derived from Eq. (36) on the condition that $a_n \neq 0$.

$$\begin{aligned} a_n' &= [\alpha \mathrm{Re}(\mu_n) + \gamma_1 \mathrm{Im}(\chi_n) \sin\theta_n - \gamma_1 \mathrm{Re}(\chi_n) \cos\theta_n] a_n \\ \theta_n' &= \sigma + 2\gamma_1 [\mathrm{Re}(\chi_n) \sin\theta_n + \mathrm{Im}(\chi_n) \cos\theta_n] - \frac{1}{2} v_1^2 \mathrm{Im}(\kappa_n) a_n^2 \end{aligned} \tag{48}$$

The Jacobian matrix of right hand function of Eq. (48) calculated at $(a_{n1,2}, \theta_{n1,2})$, is

$$\begin{bmatrix} 0 & \pm\gamma_1 \sqrt{\gamma_1^2 \mid \chi_n \mid^2 - \alpha^2 [\mathrm{Re}(\mu_n)]^2} a_{n1,2} \\ -v_1^2 \mathrm{Im}(\kappa_n) a_{n1,2} & -2\alpha \mathrm{Re}(\mu_n) \end{bmatrix} \tag{49}$$

Here the definition of nontrivial solutions

$$\begin{aligned} &\alpha \mathrm{Re}(\mu_n) + \gamma_1 \mathrm{Im}(\chi_n) \sin\theta_{n1,2} - \gamma_1 \mathrm{Re}(\chi_n) \cos\theta_{n1,2} = 0 \\ &\sigma + 2\gamma_1 [\mathrm{Re}(\chi_n) \sin\theta_{n1,2} + \mathrm{Im}(\chi_n) \cos\theta_{n1,2}] - \frac{1}{2} v_1^2 Im(\kappa_n) a_{n1,2}^2 = 0 \end{aligned} \tag{50}$$

is used. The characteristic equation of matrix (49) is

$$\lambda^2 + 2\alpha \mathrm{Re}(\mu_n) \lambda \pm \gamma_1 v_1^2 \mathrm{Im}(\kappa_n) a_{n1,2}^2 \sqrt{\gamma_1^2 \mid \chi_n \mid^2 - \alpha^2 [\mathrm{Re}(\mu_n)]^2} = 0 \tag{51}$$

According to the Routh–Hurwitz criterion, the first nontrivial solution is always stable, and the second nontrivial solution is always unstable.

5. Numerical results

Consider an axially moving beam with $v_f = 0.8$ and $\gamma_0 = 2.0$. The first two natural frequencies of unperturbed system (17) and coefficients in corresponding mode functions (21) are $\omega_1 = 5.3692$, $\beta_{11} = 3.9906$, $\beta_{21} = -1.2424 + 2.4397i$, $\beta_{31} = -1.2424 - 2.4397i$, $\beta_{41} = -1.5058$, and $\omega_2 = 30.1200$, $\beta_{12} = 7.4497$, $\beta_{22} = -1.2497 + 6.0726i$, $\beta_{32} = -1.2497 - 6.0726i$, $\beta_{42} = -4.9503$. Eq. (31) gives $\mu_1 = 45.8597$, $\chi_1 = -1.0456 + 1.1879i$, and $\mu_2 = 709.7023$, $\chi_2 = -0.4182 + 0.9776i$.

For the nonlinear model (12), Eq. (32) gives $\kappa_1 = 61.8985i$ and $\kappa_2 = 156.8368i$. Based on Eq. (39), Fig. 1 depicts the relationship between the amplitude and the detuning parameter for first two principal parametric resonance, in which the solid or dot lines stand for stable or unstable amplitudes, respectively. In the figures, $\gamma_1 = 1.0$, $k_1 = 0.2$, and $\alpha = 0.01$ (in Fig. 1(a)), 0.001 (in Fig. 1(b)). In both the first and the second principal resonance, only the trivial zero solution exists and is stable for $\sigma < \sigma_1$. At $\sigma = \sigma_1$ the trivial solution losses its stability and a stable nontrivial solution occurs. At $\sigma = \sigma_2$ the unstable trivial solution becomes stable again, and an unstable nontrivial solution bifurcates. The instability interval in the first principal resonance is larger than that in the second principal resonance, which indicates that the low order principal resonance is more significant. Fig. 2 illustrates the effect of the axial speed variation amplitude for

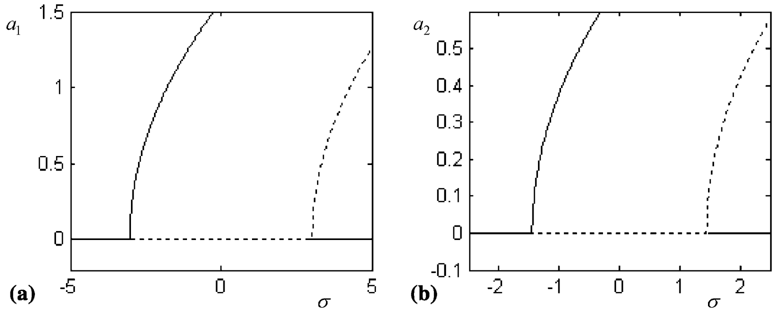

Fig. 1. The amplitude and the detuning parameter relationship of Eq. (12): (a) the first principal resonance and (b) the second principal resonance.

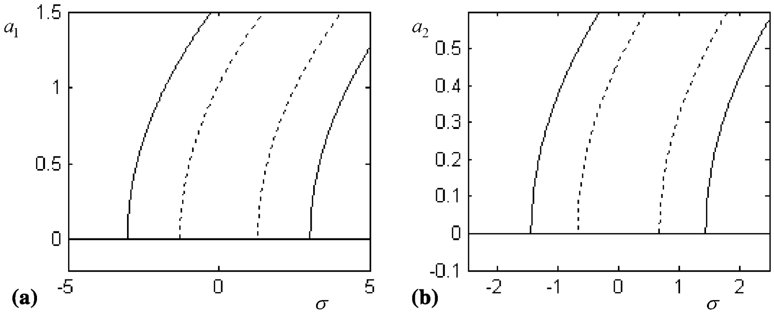

Fig. 2. The effect of the axial speed variation amplitude of Eq. (12): (a) the first principal resonance and (b) the second principal resonance.

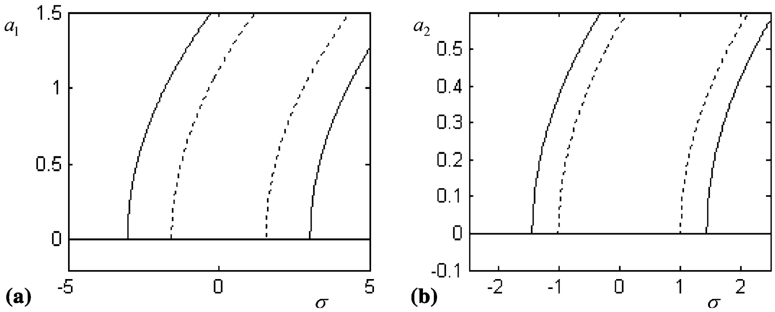

Fig. 3. The effect of viscosity of Eq. (12): (a) the first principal resonance and (b) the second principal resonance.

first two principal parametric resonance, in which the solid lines denote $\gamma_1 = 1.0$ and the dot lines denote $\gamma_1 = 0.5$ in Fig. 2(a) and $\gamma_1 = 0.8$ in Fig. 2(b). Thus the increasing speed variation amplitude leads to the larger instability interval. Fig. 3 shows the effect of viscosity coefficient, in which the solid lines denote $\alpha =$ 0.01 (in Fig. 3(a)) and $\alpha = 0.001$ (in Fig. 3(b)) and the dot lines denote $\alpha = 0.03$ (in Fig. 3(a)) and $\alpha = 0.0012$ (in Fig. 3(b)). Hence the larger viscosity coefficient results in the smaller instability interval. Fig. 4 displays the effect of nonlinearity, in which the solid and dot lines, respectively stand for $k_1 = 0.2$ and $k_1 = 0.25$. The amplitudes of both the stable and unstable nontrivial solutions increase with the decrease of the nonlinear term coefficient, while the instability interval is independent of the coefficient.

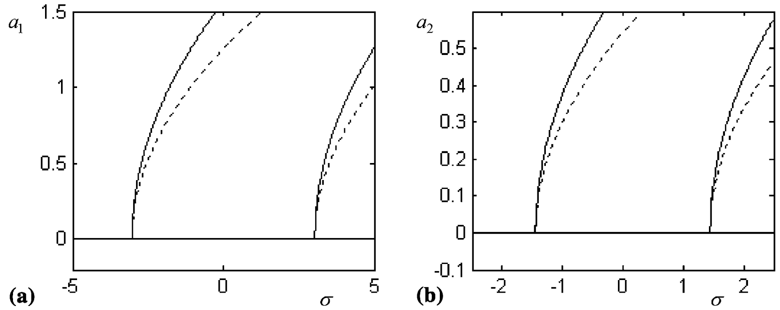

Fig. 4. The effect of nonlinearity of Eq. (12): (a) the first principal resonance and (b) the second principal resonance.

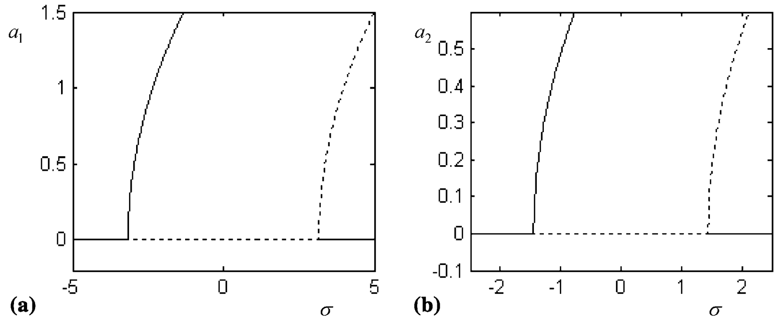

Fig. 5. The amplitude and the detuning parameter relationship of Eq. (13): (a) the first principal resonance and (b) the second principal resonance.

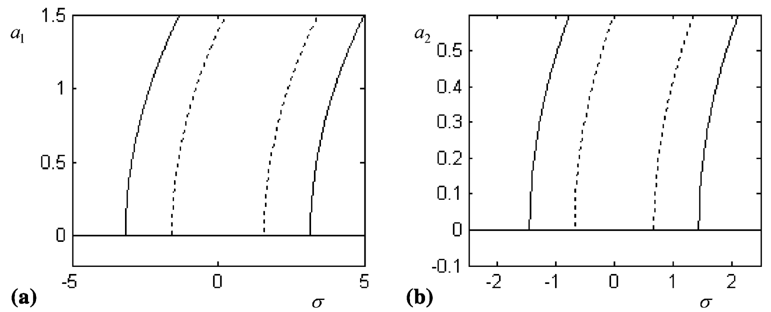

Fig. 6. The effect of the axial speed variation amplitude of Eq. (13): (a) the first principal resonance and (b) the second principal resonance.

For the nonlinear model (13), Eq. (33) gives $\kappa_1 = 40.9617i$ and $\kappa_2 = 94.4142i$. Fig. 5 demonstrates the relationship between the amplitude and the detuning parameter for first two principal parametric resonance, and Figs. 6–8 present the effects of the speed variation, the viscosity coefficient, and the nonlinear term coefficient, respectively. In these figures, all parameters are the same as those in corresponding figures for nonlinear model (12). The numerical simulations indicate that two models are qualitatively same, while there exist quantitative differences. In fact, Fig. 9 is the superposition of Figs. 1 and 5, in which the solid and dot lines represent the results of Eqs. (12) and (13). The nontrivial solution amplitude derived from Eq. (12) is smaller, and the instability intervals are the same.

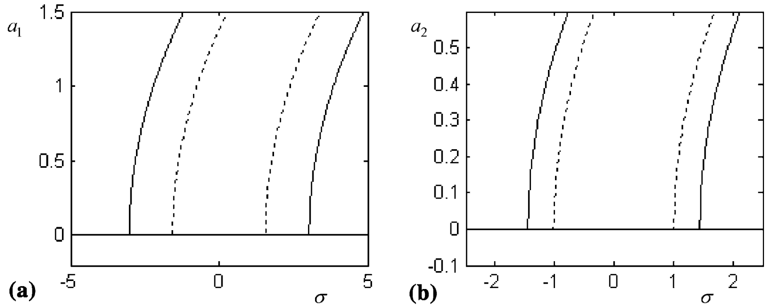

Fig. 7. The effect of viscosity of Eq. (13): (a) the first principal resonance and (b) the second principal resonance.

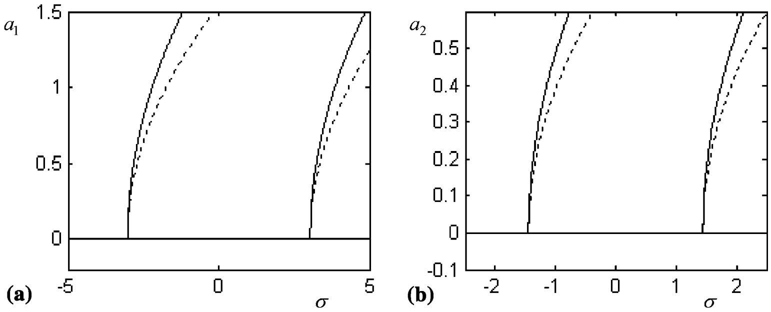

Fig. 8. The effect of nonlinearity of Eq. (13): (a) the first principal resonance and (b) the second principal resonance.

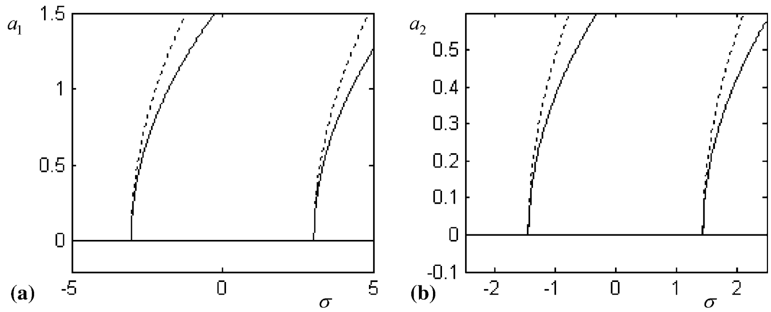

Fig. 9. Comparison of two nonlinear models: (a) the first principal resonance and (b) the second principal resonance.

6. Conclusions

This paper is devoted to nonlinear parametric vibration of axially accelerating viscoelastic beams. A nonlinear partial-differential equation governing transverse vibration is derived from the Newton second law, the Kelvin constitutive relationship, and the Lagrangian strain. Under the assumption that the tension of beam can be replaced by the averaged tension over the beam, the equation reduces an integro-partial-differential equation. The two equations are analyzed via the method of multiple scales in principal parametric resonance. The nontrivial steady-state response and its existence conditions are presented. The Lyapunov linearized stability theory is applied to obtain the stability conditions of straight equilibrium and nontrivial steady-state response. The investigation demonstrates that there exists a instability interval of the detuning parameters on which the straight equilibrium is unstable, and the first (second) nontrivial steady-state response is always stable (unstable). Numerical calculations show that the models have the same tendencies to change with related parameters. The lower order principal resonance has the larger instability interval. The instability interval increases with the increasing axial speed variation amplitude and the decreasing viscosity coefficient. The amplitude of nontrivial solutions increases with the decrease of the nonlinear term coefficient. Although both models yields the same instability interval, the amplitude of nontrivial solutions calculated from two models are slightly different.

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References

Abrate, A.S., 1992. Vibration of belts and belt drives. Mechanism and Machine Theory 27 (6), 645–659.

Chakraborty, G., Mallik, A.K., 1998. Parametrically excited nonlinear traveling beams with and without external forcing. Nonlinear Dynamics 17, 301–324.

Chakraborty, G., Mallik, A.K., Hatwal, H., 1999. Non-linear vibration of a travelling beam. International Journal of Non-Linear Mechanics 34, 655–670.

Chen, L.Q., in press. Analysis and control of transverse vibrations of axially moving strings. ASME Applied Mechanics Reviews.

Chen, L.Q., Zhang, N.H., 2004. Nonlinear dynamics of axially moving viscoelastic strings based on translating eigenfunctions. XXI International Congress of Theoretical and Applied Mechanics, Warsaw, Poland, 15–21 August 2004.

Chen, L.Q., Zhao, W.J., in press. A computation method for nonlinear vibration of axially accelerating viscoelastic strings. Applied Mathematics and Computation.

Chen, L.Q., Zu, J.W., 2003. Parametrical resonance of the excited axially moving string with an integral constitutive law. International Journal of Nonlinear Science and Numerical Simulation 4 (2), 169–177.

Chen, L.Q., Wu, J., Zu, J.W., 2004a. Asymptotic nonlinear behaviors in transverse vibration of an axially accelerating viscoelastic string. Nonlinear Dynamics 35 (4), 347–360.

Chen, L.Q., Wu, J., Zu, J.W., 2004b. The chaotic response of the viscoelastic traveling string: an integral constitutive law. Chaos, Solitons and Fractals 21 (2), 349–357.

Chen, L.Q., Yang, X.D., Cheng, C.J., 2004c. Dynamic stability of an axially accelerating viscoelastic beam. European Journal of Mechanics A/Solid 23 (4), 659–666.

Chen, L.Q., Zu, J.W., Wu, J., Yang, X.D., 2004d. Transverse vibrations of an axially accelerating viscoelastic string with geometric nonlinearity. Journal of Engineering Mathematics 48 (2), 171–182.

Chen, L.Q., Zhao, W.J., Zu, J.W., in press. Transient responses of an axially accelerating viscoelastic string constituted by a fractional differentiation law. Journal of Sound and Vibration.

Chen, L.Q., Zhang, N.H., Zu, J.W., 2003. The regular and chaotic vibrations of an axially moving viscoelastic string based on 4-order Galerkin truncation. Journal of Sound and Vibration 261 (4), 764–773.

Fung, R.F., Huang, J.S., Chen, Y.C., 1997. The transient amplitude of the viscoelastic travelling string: an integral constitutive law. Journal of Sound and Vibration 201 (2), 153–167.

Fung, R.F., Huang, J.S., Chen, Y.C., Yao, C.M., 1998. Nonlinear dynamic analysis of the viscoelastic string with a harmonically varying transport speed. Computers and Structures 66 (6), 777–784.

Marynowski, K., 2002. Non-linear dynamic analysis of an axially moving viscoelastic beam. Journal of Theoretical and Applied Mechanics 40, 465–482.

Marynowski, K., 2004. Non-linear vibrations of an axially moving viscoelastic web with time-dependent tension. Chaos, Solitons, and Fractals 21, 481–490.

Marynowski, K., Kapitaniak, T., 2002. Kelvin–Voigt versus Bügers internal damping in modeling of axially moving viscoelastic web. International Journal of Non-Linear Mechanics 37, 1147–1161.

Öz, H.R., 2001. On the vibrations of an axially traveling beam on fixed supports with variable velocity. Journal of Sound and Vibration 239, 556–564.

Öz, H.R., Pakdemirli, M., 1999. Vibrations of an axially moving beam with time dependent velocity. Journal of Sound and Vibration 227, 239–257.

Öz, H.R., Pakdemirli, M., Boyaci, H., 2001. Non-linear vibrations and stability of an axially moving beam with time-dependent velocity. International Journal Non-Linear Mechanics 36, 107–115.

Öz, H.R., Pakdemirli, M., Özkaya, E., 1998. Transition behaviour from string to beam for an axially accelerating material. Journal of Sound Vibration 215, 571–576.

Özkaya, E., Öz, H.R., 2002. Determination of natural frequencies and stability regions of axially moving beams using artificial neural networks method. Journal of Sound Vibration 254, 782–789.

Özkaya, E., Pakdemirli, M., 2000. Vibrations of an axially accelerating beam with small flexural stiffness. Journal of Sound and Vibration 234, 521–535.

Pakdemirli, M., Ulsoy, A.G., 1997. Stability analysis of an axially accelerating string. Journal of Sound and Vibration 203, 815–832.

Pakdemirli, M., Ulsoy, A.G., Ceranoglu, A., 1994. Transverse vibration of an axially accelerating string. Journal of Sound and Vibration 169, 179–196.

Park, S.W., 2001. Analytical modeling of viscoelastic dampers for structural and vibration control. International Journal of Solids and Structures 38, 8065–8092.

Parker, R.G., Lin, Y., 2001. Parametric instability of axially moving media subjected to multifrequency tension and speed fluctuations. Journal of Applied Mechanics, ASME 68 (1), 49–57.

Pasin, F., 1972. Ueber die stabilität der beigeschwingungen von in laengsrichtung periodisch hin und herbewegten stäben. Ingenieur-Archiv 41, 387–393.

Pellicano, F., Vestroni, F., 2000. Nonlinear dynamics and bifurcations of an axially moving beam. Journal of Vibration and Acoustics 122, 21–30.

Pellicano, F., Vestroni, F., 2002. Complex dynamic of high-speed axially moving systems. Journal of Sound and Vibration 258, 31–44.

Pellicano, F., Zirilli, F., 1997. Boundary layers and non-linear vibrations in an axially moving beam. International Journal of Non-Linear Mechanics 33, 691–711.

Pellicano, F., Fregolent, A., Bertuzzi, A., Vestroni, F., 2001. Primary and parametric non-linear resonances of a power transmission belt. Journal of Sound and Vibration 244, 669–684.

Ravindra, B., Zhu, W.D., 1998. Low dimensional chaotic response of axially accelerating continuum in the supercritical regime. Achieve of Applied Mechanics 68, 195–205.

Wickert, J.A., 1992. Non-linear vibration of a traveling tensioned beam. International Journal of Non-Linear Mechanics 27, 503–517.

Wickert, J.A., Mote Jr., C.D., 1988. Current research on the vibration and stability of axially-moving materials. The Shock and Vibration Digest 20 (5), 3–13.

Wickert, J.A., Mote Jr., C.D., 1990. Classical vibration analysis of axially moving continua. Journal of Applied Mechanics, ASME 57, 738–744.

Yang, X.D., Chen, L.Q., 2005. Bifurcation and chaos of an axially accelerating viscoelastic beam. Chaos, Solitons and Fractals 23, 249–258.

Zhang, L., Zu, J.W., 1999a. Nonlinear vibration of parametrically excited moving belts, part 1: dynamic response. Journal of Applied Mechanics, ASME 66 (2), 396–402.

Zhang, L., Zu, J.W., 1999b. Nonlinear vibration of parametrically excited moving belts, part 2: stability analysis. Journal of Applied Mechanics, ASME 66 (2), 403–409.

Zhao, W.J., Chen, L.Q., 2002. A numerical algorithm for non-linear parametric vibration analysis of a viscoelastic moving belt. International Journal of Nonlinear Science and Numerical Simulation 3 (2), 139–144.